JOURNAL OF APPROXIMATION THEORY 29, 278–293 (1980)

A Natural Interpolation of C^K Functions

PAUL KERGIN*

Department of Mathematics, University of Toronto, Toronto, Canada

Communicated by Carl de Boor

Received December 28, 1978

DEDICATED TO THE MEMORY OF P. TURÁN

1. INTRODUCTION

This paper gives a linear projection χ from the space of K times continuously differentiable functions on $\mathbb{R}^N$ onto the space of polynomials of degree $\leqslant K$ on $\mathbb{R}^N$. The projection depends only on $K + 1$ fixed points $x_0, ..., x_K \in \mathbb{R}^N$.

When $N = 1$, $\chi(f)$ is the Hermite interpolating polynomial of f at $x_0, ..., x_K \in \mathbb{R}$, i.e., the polynomial p of degree $\leqslant K$ for which $f - p$ vanishes at $x_0, ..., x_K$, with repetitions in the x_i giving rise to a multiple zero of $f - p$ in the usual way. One method of computing p is solving a system of $K + 1$ linear equations in $K + 1$ unknowns. The unknowns are the coefficients of p and the equations are $p(x_k) = f(x_k)$ $(k = 0, ..., K)$ if $x_0, ..., x_K$ are distinct.

A straightforward generalization of the Hermite interpolation problem to $\mathbb{R}^N$ $(N \geqslant 2)$ can be stated: if $x_0, ..., x_K$ are points in $\mathbb{R}^N$ and $f: \mathbb{R}^N \to \mathbb{R}$, find a polynomial $p: \mathbb{R}^N \to \mathbb{R}$, of degree $\leqslant K$, such that $(p - f)(x_k) = 0$ for all k. Again, when $x_0, ..., x_K$ are distinct, p can be found by solving a system of $K + 1$ linear equations, but now there are $\binom{K+N}{K}$ unknowns, and p is no longer uniquely determined if $K \geqslant 1$.

Additional conditions must be imposed on p to ensure that it is unique. Glaeser, in his "schemes of interpolation" given in [4], requires that p lie in a $K + 1$-dimensional subspace of the space of polynomials of degree $\leqslant K$. The choice of subspace is arbitrary among those subspaces of dimension $K + 1$ which contain a solution of the linear system $p(x_k) = f(x_k)$ $(k = 0, ..., K)$ for all f. The only information needed to find p is the values of f at $x_0, ..., x_K$.

Here, we impose the conditions: p must depend linearly on f and, if $q(\partial/\partial x)$ is a constant coefficient differential operator having terms all of the same order $k \in \{0, ..., K\}$, then $q(\partial/\partial x)\,(p - f)$ must equal zero at some point in the con-

* Supported in part by University of Toronto Open Scholarships and by the National Research Council of Canada.

0021-9045/80/080278–16$02.00/0

Copyright © 1980 by Academic Press, Inc.
All rights of reproduction in any form reserved.

vex hull of any $k+1$ of the points $x_0, ..., x_K$. Thus, if f is a solution of the equation $q(\partial/\partial x)(f) \equiv 0$, so is p.

The proof of the existence of $\chi: f \mapsto p$ given here is an application of Stokes' theorem. For example, let $K = 1$ and $x_0 = (0, 0)$, $x_1 = (1, 0) \in \mathbb{R}^2$. If f is continuously differentiable, we require that $p(x_0) = f(x_0)$, $p(x_1) = f(x_1)$, and $\int_0^1 (\partial p/\partial y)(x, 0)\, dx = \int_0^1 (\partial f/\partial y)(x, 0)\, dx$. Let $q(\partial/\partial x, \partial/\partial y) = a(\partial/\partial x) + b(\partial/\partial y)$, where $a, b \in \mathbb{R}$. Then

$$\int_0^1 q\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)(p-f)(x,0)\,dx$$
$$= a((p-f)(1,0) - (p-f)(0,0))$$
$$+ b\int_0^1 \frac{\partial}{\partial y}(p-f)(x,0)\,dx$$

by the mean value theorem, and this equals zero. Therefore, there exists $x \in [0, 1]$ such that $q(\partial/\partial x, \partial/\partial y)(p-f)(x, 0) = 0$.

Micchelli and Milman [6] give a proof of the existence of χ by exhibiting an explicit expression for $\chi(f)$, analogous to Newton's form of Hermite interpolation.

Sections 3 and 4 prove the uniqueness, existence, and some properties of χ. Section 5 gives versions of χ for complex analytic functions and real differential forms. Section 6 is an application to a problem of convergence of distributions and Section 7 contains a formula not involving integrals for calculating $\chi(f)$ when f is a polynomial.

2.

We use the following notation: $\mathbb{N} = \{0, 1, 2, ...\}$ and $\mathbb{N}^+ = \{1, 2, ...\}$. If $N \in \mathbb{N}^+$, $\mathbb{R}_N$ and $\mathbb{C}_N$ represent the algebraic duals of $\mathbb{R}^N$ and $\mathbb{C}^N$. The dimension of a finite-dimensional vector space E over $\mathbb{R}$ is written $\dim E$.

If J is a finite set, card J is the number of elements in J. An indexed subset of $\mathbb{R}^N$, $\{x_j\}_{j \in J}$, has convex hull $[x_j]_{j \in J}$.

For a polynomial $p: \mathbb{R}^N \to \mathbb{R}$ or $p: \mathbb{C}^N \to \mathbb{C}$, $\deg p$ is its degree. If $K \in \mathbb{N}$, $P^K(\mathbb{R}^N)$ and $P^K(\mathbb{C}^N)$ are the Hausdorff vector spaces of real and complex polynomials of degree $\leqslant K$.

A multi-index is an element of $\mathbb{N}^N$. If

$$j = (j_1, ..., j_N) \in \mathbb{N}^N, \qquad |j| = j_1 + \cdots + j_N \qquad \text{and} \qquad j! = j_1! \cdots j_N!.$$

If also $i \in \mathbb{N}^N$, then $i \leqslant j$ whenever $i_n \leqslant j_n$ for all n. In this case, $\binom{j}{i} = j!/i!(j-i)!$. If $x \in \mathbb{R}^N$ or $\mathbb{C}^N$, then $x^j = x_1^{j_1} \cdots x_N^{j_N}$. $\partial^{|j|}/\partial x^j$ is the differential operator on $\mathbb{R}^N$, $\partial^{|j|}/(\partial x_1^{j_1} \cdots \partial x_N^{j_N})$. $Q^K(\mathbb{R}^N)$ is the real vector space of constant coefficient differential operators which are homogeneous of order K. That is, an element q of $Q^K(\mathbb{R}^N)$ is a linear combination of operators $\partial^{|j|}/\partial x^j$ such that $|j| = K$.

$\mathscr{C}^K(\mathbb{R}^N)$ is the vector space of K-times continuously differentiable functions $f: \mathbb{R}^N \to \mathbb{R}$. It has the topology of uniform convergence of derivatives of order $\leqslant K$ on compact sets, given by the family of seminorms

$$f \mapsto \max_{\substack{|k| \leqslant K \\ x \in X}} \left| \frac{\partial^{|k|}}{\partial x^k} f(x) \right|,$$

where X ranges over all compact subsets of $\mathbb{R}^N$. $\mathscr{D}^K(\mathbb{R}^N)$ is the vector space of continuous linear functionals on $\mathscr{C}^K(\mathbb{R}^N)$.

3.

This section proves uniqueness of the map χ of the following theorem. Existence is proven in Section 4.

THEOREM 3.1. *Let $N \in \mathbb{N}^+$, $K \in \mathbb{N}$, and $x_0, \ldots, x_K \in \mathbb{R}^N$, not necessarily distinct. There is a unique $\chi: \mathscr{C}^K(\mathbb{R}^N) \to P^K(\mathbb{R}^N)$ satisfying*:

(3.2) *χ is linear.*

(3.3) *for every $f \in \mathscr{C}^K(\mathbb{R}^N)$, every $q \in Q^k(\mathbb{R}^N)$, where $k \in \{0, \ldots, K\}$, and every $J \subset \{0, \ldots, K\}$ with* card $J = k + 1$, *there exists $x \in [x_j]_{j \in J}$ such that $q(\partial/\partial x)(\chi(f) - f)(x) = 0$.*

Remark 3.4. Let y be one of the points $x_0, \ldots, x_K$, let $J = \{j \in \{0, \ldots, K\}$ s.t. $x_j = y\}$, and let $m \in \{0, \ldots, \text{card } J - 1\}$. Property (3.3) implies, for every $f \in \mathscr{C}^K(\mathbb{R}^N)$ and $i \in \mathbb{N}^N$ with $|i| = m$, that there exists $x \in [x_j]_{j \in J}$ such that $(\partial^{|i|}/\partial x^i)(\chi(f) - f)(x) = 0$. That is, $\chi(f) - f$ is flat of order card $J - 1$ at y. If $N = 1$, this is precisely the property that characterizes the Hermite interpolating polynomial of f because $\deg \chi(f) \leqslant K$. See Wendroff [8, Chapter 1].

For all N, $\chi(f)$ is indeed an interpolation of f at $x_0, \ldots, x_K$ because $(\chi(f) - f)(x_j) = 0$ for all j.

PROPOSITION 3.5. *If $\chi: \mathscr{C}^K(\mathbb{R}^N) \to P^K(\mathbb{R}^N)$ satisfies* (3.2) *and* (3.3), *it is continuous.*

Proof. Choose $\delta > 0$ so small that, if $p \in P^K(\mathbb{R}^N)$ satisfies the property that, for every $i \in \mathbb{N}^N$ with $|i| \leqslant K$, there exists $x \in [x_j]_{j \in \{0, \ldots, K\}}$ (depending on i) such that $|(\partial^{|i|}/\partial x^i)\, p(x)| \leqslant \delta$, then p also satisfies the property that each of its coefficients is in absolute value $\leqslant 1$. By (3.3) applied to the differential operators $\partial^{|i|}/\partial x^i$, if $f \in \mathscr{C}^K(\mathbb{R}^N)$ such that

$$\max_{\substack{i \in \mathbb{N}^N, |i| \leqslant K \\ x \in [x_j]_{j \in \{0,\ldots,K\}}}} \left\{ \left| \frac{\partial^{|i|}}{\partial x^i} f(x) \right| \right\} \leqslant \delta$$

then the absolute value of each coefficient of $\chi(f)$ is $\leqslant 1$.

Remark 3.6. The next proposition shows that if a function f in $\mathscr{C}^K(\mathbb{R}^N)$ is a solution of the differential equation $q(\partial/\partial x) f(x) \equiv 0$, where q is a homogeneous constant coefficient operator, then $\chi(f)$ is also a solution. In particular, if f is zero along a constant vector field, so is $\chi(f)$.

PROPOSITION 3.7. *Let $K \in \mathbb{N}$ and suppose $\chi: \mathscr{C}^K(\mathbb{R}^N) \to P^K(\mathbb{R}^N)$ satisfies (3.3). Let $q \in Q^k(\mathbb{R}^N)$, where $k \in \{0,\ldots, K\}$ and $f \in \mathscr{C}^K(\mathbb{R}^N)$ such that $q(\partial/\partial x)(f)$ is identically zero. Then $q(\partial/\partial x)(\chi(f))$ is identically zero.*

Proof. Let $\chi(f) = p_0 + \cdots + p_K$ be the homogeneous decomposition of $\chi(f)$. It suffices to show for each $l \in \{k,\ldots, K\}$, that $q(\partial/\partial x)(p_l)$ is identically zero. We use decreasing induction. Fix $L \in \{k,\ldots, K\}$ and make the inductive hypothesis that $q(\partial/\partial x)(p_l)$ is identically zero for $l > L$. The hypothesis is trivial if $L = K$.

By property (3.3), for each $i \in \mathbb{N}^N$ with $|i| = L - k$, there exists $x \in \mathbb{R}^N$ such that

$$\frac{\partial^{|i|}}{\partial x^i} \cdot q\left(\frac{\partial}{\partial x}\right) \chi(f)(x) = \frac{\partial^{|i|}}{\partial x^i} \cdot q\left(\frac{\partial}{\partial x}\right) f(x) = 0.$$

Therefore,

$$\frac{\partial^{|i|}}{\partial x^i} \cdot q\left(\frac{\partial}{\partial x}\right) p_L(x) = \frac{\partial^{|i|}}{\partial x^i} \cdot q\left(\frac{\partial}{\partial x}\right) (p_0 + \cdots + p_K)(x) = 0,$$

because for $l < L$, $\deg p_l <$ the order of $(\partial^{|i|}/\partial x^i) \cdot q(\partial/\partial x)$ and by the inductive hypothesis for $l > L$.

But $q(\partial/\partial x)(p_L)$ is homogeneous of degree $L - k$ and each of its derivatives of order $L - k$ is zero at some point in $\mathbb{R}^N$, so it is identically zero.

Remark 3.8. Proposition 3.9 shows, if $f \in \mathscr{C}^K(\mathbb{R}^N)$ is constant on each hyperplane in $\mathbb{R}^N$ which is parallel to some fixed hyperplane, then $\chi(f)$ is also constant on each of the hyperplanes.

PROPOSITION 3.9. *Let $K \in \mathbb{N}$ and suppose $\chi: \mathscr{C}^K(\mathbb{R}^N) \to P^K(\mathbb{R}^N)$ satisfies (3.2) and (3.3). Let $\lambda \in \mathbb{R}_N$ be a linear functional on $\mathbb{R}^N$ and let $f \in \mathscr{C}^K(\mathbb{R})$. Then $\chi(f \circ \lambda) = \psi(f) \circ \lambda$, where $\psi(f)$ is the Hermite interpolating polynomial of f at $\lambda(x_0),\ldots, \lambda(x_K)$.*

Proof. We may assume $\lambda \neq 0$.

Let $g: \mathbb{R} \to \mathbb{R}$ be given by $(g \circ \lambda)(x) = \chi(f \circ \lambda)(x)$. To show g is well-defined, fix w and $x \in \mathbb{R}^N$ such that $\lambda(w) = \lambda(x)$. Let $q(\partial/\partial x) = \sum_{n=1}^{N} v_n \times \partial/\partial x_n \in Q^1(\mathbb{R}^N)$, where v_n is the nth component of $w - x$. Since $q(\partial/\partial x)(\lambda)$ is identically zero on $\mathbb{R}^N$, so is $q(\partial/\partial x)(f \circ \lambda)$ if $K \in \mathbb{N}^+$. Therefore $q(\partial/\partial x)(\chi(f \circ \lambda))$ is identically zero, by Proposition 3.7 if $K \in \mathbb{N}^+$ and because $\deg \chi(f \circ \lambda) = 0$ if $K = 0$. In particular, $\chi(f \circ \lambda)$ is constant on the line segment joining x to w.

Now $g \in P^K(\mathbb{R})$. For, choose any linear $\gamma: \mathbb{R} \to \mathbb{R}^N$ such that $\lambda \circ \gamma$ is the identity on $\mathbb{R}$. Then $g = \chi(f \circ \lambda) \circ \gamma$.

Let $y \in \mathbb{R}$ be any one of the points $\lambda(x_0), \ldots, \lambda(x_K)$, let $J = \{j \in \{0, \ldots, K\}$ s.t. $\lambda(x_j) = y\}$, and let $k = \operatorname{card} J - 1$. By Remark 3.4, it only remains to show that $g - f$ is flat of order k at y.

Fix $m \in \{0, \ldots, k\}$ and choose $n \in \{1, \ldots, N\}$ such that $(\partial/\partial x_n)\lambda \neq 0$. By property (3.3), there exists $x \in [x_j]_{j \in J}$ such that $(\partial^m/\partial x_n{}^m)(\chi(f \circ \lambda) - (f \circ \lambda))(x) = 0$. Since $g \circ \lambda = \chi(f \circ \lambda)$, $(\partial^m/\partial x_n{}^m)((g - f) \circ \lambda)(x) = 0$. Applying the chain rule, $(d^m/dy^m)(g - f)(\lambda(x)) \cdot (\partial\lambda/\partial x_n)^m = 0$. But $x \in [x_j]_{j \in J}$ and $\lambda(x_j) = y$ for all $j \in J$, so $\lambda(x) = y$ and $(d^m/dy^m)(g - f)(y) = 0$.

COROLLARY 3.10. *χ is unique.*

Proof. Since the polynomials are a dense linear subspace of $\mathscr{C}^K(\mathbb{R}^N)$ (see Treves [7, p. 160]) and χ is continuous (Proposition 3.5) it suffices to show that the restriction of χ to the space of polynomials in $\mathbb{R}^N$ is unique. But every polynomial in $\mathbb{R}^N$ can be written as a sum of polynomials of the form $p \circ \lambda$, where $\lambda \in \mathbb{R}_N$ and p is a polynomial in $\mathbb{R}$. By Proposition 3.9, $\chi(p \circ \lambda) = \psi(p) \circ \lambda$ and the corollary follows by linearity of χ.

4.

The existence of χ is proven here by defining a certain subspace of $\mathscr{D}^K(\mathbb{R}^N)$, showing that the dimension of this subspace is $\binom{K+N}{K} = \dim P^K(\mathbb{R}^N)$, and requiring, for each of its elements T and each $f \in \mathscr{C}^K(\mathbb{R}^N)$, that $T(\chi(f) - f) = 0$.

Remark 4.1. The following notation for differential forms is used in this section and Section 5. For $N \in \mathbb{N}^+$ and $k, m \in \mathbb{N}$, let $A^{k,m}(\mathbb{R}^N)$ be the vector space of differential k forms on $\mathbb{R}^N$ which are m times continuously differentiable. An element ω of $A^{k,m}(\mathbb{R}^N)$ will be written

$$\omega = \sum_{1 \leqslant s_1 < \cdots < s_k \leqslant N} f_{s_1 \cdots s_k} \lambda_{s_1} \wedge \cdots \wedge \lambda_{s_k},$$

where $\lambda_1, \ldots, \lambda_N$ is a basis of $\mathbb{R}_N$ and $f_{s_1 \cdots s_k} \in \mathscr{C}^m(\mathbb{R}^N)$. For $m \in \mathbb{N}^+$,

d: $A^{k,m}(\mathbb{R}^N) \to A^{k+1,m-1}(\mathbb{R}^N)$ is the exterior derivative, given by

$$d\omega = \sum_{1 \leqslant s_1 < \cdots < s_k \leqslant N} \sum_{n=1}^{N} \frac{\partial}{\partial y_n} f_{s_1 \cdots s_k} \lambda_n \wedge \lambda_{s_1} \wedge \cdots \wedge \lambda_{s_k},$$

where $y_1, \ldots, y_N \in \mathbb{R}^N$ is the dual basis of $\lambda_1, \ldots, \lambda_N$, and $\partial/\partial y_n = \sum_{l=1}^{N} y_{nl} \times (\partial/\partial x_l) \in Q^1(\mathbb{R}^N)$, y_{nl} being the lth component of y_n.

Similarly, let $G^k(\mathbb{R}^N)$ denote the vector space, $Q^k(\mathbb{R}^N) \otimes$ the kth exterior product of $\mathbb{R}_N$. An element μ of $G^k(\mathbb{R}^N)$ will be written

$$\mu = \sum_{1 \leqslant s_1 < \cdots < s_k \leqslant N} q_{s_1 \cdots s_k}\left(\frac{\partial}{\partial x}\right) \lambda_{s_1} \wedge \cdots \wedge \lambda_{s_k},$$

where $q_{s_1 \ldots s_k} \in Q^k(\mathbb{R}^N)$. Define δ: $G^k(\mathbb{R}^N) \to G^{k+1}(\mathbb{R}^N)$ by

$$\delta(\mu) = \sum_{1 \leqslant s_1 < \cdots < s_k \leqslant N} \sum_{n=1}^{N} \frac{\partial}{\partial y_n} \cdot q_{s_1 \cdots s_k}\left(\frac{\partial}{\partial x}\right) \lambda_n \wedge \lambda_{s_1} \wedge \cdots \wedge \lambda_{s_k}.$$

LEMMA 4.2. *Let $N \in \mathbb{N}^+$, $K \in \mathbb{N}$, and $x_0, \ldots, x_K \in \mathbb{R}^N$ independent (that is, if $a_k \in \mathbb{R}$ for $k \in \{0, \ldots, K\}$ with $\sum_{k=0}^{K} a_k = 0$ and $\sum_{k=0}^{K} a_k x_k = 0$, then $a_k = 0$ for all k).*

For each $k \in \{0, \ldots, K\}$, let

$$\phi_k : G^k(\mathbb{R}^N) \times \mathscr{C}^K(\mathbb{R}^N) \to A^{k,K-k}(\mathbb{R}^N)$$

be the bilinear map given by

$$\phi_k\Big(\sum_{1 \leqslant s_1 < \cdots < s_k \leqslant N} q_{s_1 \cdots s_k}\left(\frac{\partial}{\partial x}\right) \lambda_{s_1} \wedge \cdots \wedge \lambda_{s_k}, f\Big) = \sum_{1 \leqslant s_1 < \cdots < s_k \leqslant N} q_{s_1 \cdots s_k}\left(\frac{\partial}{\partial x}\right)(f) \lambda_{s_1} \wedge \cdots \wedge \lambda_{s_k}.$$

(ϕ_k is independent of the choice of basis $\lambda_1, \ldots, \lambda_N$ of $\mathbb{R}_N$.)

For $J \subset \{0, \ldots, K\}$ with card $J = k + 1$, *let $B_J = \{T \in \mathscr{D}^K(\mathbb{R}^N)$* s.t. *there exists $\mu \in G^k(\mathbb{R}^N)$ such that $T(f) = \int_{[x_j]_{j \in J}} \phi_k(\mu, f)$ for every $f \in \mathscr{C}^K(\mathbb{R}^N)\}$. Also, let $B_\phi = \{0\} \subset \mathscr{D}^K(\mathbb{R}^N)$.*

Then $\dim \sum_{J \subset \{0, \ldots, K\}} B_J \leqslant \binom{K+N}{K}$.

(Equality is proven in Corollary 4.5.)

Proof. We show first, for each non-empty $J \subset \{0, \ldots, K\}$, that

$$\dim\Big(B_J / B_J \cap \sum_{\substack{I \subset J \\ \operatorname{card} I = k}} B_I\Big) \leqslant \binom{N-1}{k},$$

where $k + 1 = \operatorname{card} J$. If $k = 0$, B_J is the one-dimensional subspace of $\mathscr{D}^K(\mathbb{R}^N)$ generated by the Dirac measure at some x_j, so suppose that $k \in \{1,..., K\}$.

Choose an orientation for $[x_j]_{j\in J}$ and for every $I \subset J$ with card $I = k$, give $[x_i]_{i\in I}$ the orientation induced by that of $[x_j]_{j\in J}$. Let $\psi: G^k(\mathbb{R}^N) \to B_J$ be given by

$$\psi(\mu)(f) = \int_{[x_j]_{j\in J}} \phi_k(\mu, f)$$

for $f \in \mathscr{C}^K(\mathbb{R}^N)$. ψ is linear and onto by the definition of B_J.

Let $C \subset \mathbb{R}^N$ be the subspace generated by $\{x_j - x_i \text{ s.t. } i, j \in J\}$ so that $\dim C = k$, by the independence of $\{x_j\}_{j\in J}$. Fix $y_1,..., y_k$, a basis of C, complete it to a basis $y_1,..., y_k, y_{k+1},..., y_N$ of $\mathbb{R}^N$, and let $\lambda_1,..., \lambda_N$ be the dual basis. Also let

$$E = \Big\{ \sum_{1\leqslant s_1<\cdots<s_k\leqslant N} q_{s_1\cdots s_k}\left(\frac{\partial}{\partial x}\right) \lambda_{s_1} \wedge \cdots \wedge \lambda_{s_k} \in G^k(\mathbb{R}^N) \text{ s.t.}$$

$$q_{12\cdots k}\left(\frac{\partial}{\partial x}\right) \text{ is of the form } \frac{\partial}{\partial y_1} \cdot r_1\left(\frac{\partial}{\partial x}\right) + \cdots + \frac{\partial}{\partial y_k} \cdot r_k\left(\frac{\partial}{\partial x}\right)$$

$$\text{where } r_1,..., r_k \in Q^{k-1}(\mathbb{R}^N)\Big\}.$$

We will show for each $\mu \in E$, that

$$\psi(\mu) \in \sum_{\substack{I\subset J\\ \operatorname{card} I=k}} B_I.$$

For, fix $r_1,..., r_k \in Q^{k-1}(\mathbb{R}^N)$ such that $q_{12...k}$, the $\lambda_1 \wedge \cdots \wedge \lambda_k$ term of μ, equals $\sum_{l=1}^k (\partial/\partial y_l) \cdot r_l(\partial/\partial x)$. Let $\nu \in G^{k-1}(\mathbb{R}^N)$ be given by

$$\nu = \sum_{l=1}^{k} (-1)^{l+1} r_l\left(\frac{\partial}{\partial x}\right) \lambda_1 \wedge \cdots \wedge \lambda_{l-1} \wedge \lambda_{l+1} \wedge \cdots \wedge \lambda_k.$$

The $\lambda_1 \wedge \cdots \wedge \lambda_k$ term of $\delta\nu$ equals $q_{12...k}$. Therefore

$$\int_{[x_j]_{j\in J}} \phi_k(\mu, f) = \int_{[x_j]_{j\in J}} \phi_k(\delta\nu, f)$$

for all $f \in \mathscr{C}^K(\mathbb{R}^N)$ because $\lambda_l(C) = 0$ for $l \in \{k + 1,..., N\}$.

A straightforward calculation using the definitions of ϕ_k, δ, and d shows that $\phi_k(\delta\nu, f) = d\phi_{k-1}(\nu, f)$ for all $f \in \mathscr{C}^K(\mathbb{R}^N)$. By Stokes' theorem for Euclidean simplices,

$$\int_{[x_j]_{j\in J}} \phi_k(\mu, f) = \int_{[x_j]_{j\in J}} d\phi_{k-1}(\nu, f) = \sum_{\substack{I\subset J\\ \operatorname{card} I=k}} \int_{[x_i]_{i\in I}} \phi_{k-1}(\nu, f).$$

Therefore

$$\psi(\mu) \in \sum_{\substack{I \subset J \\ \operatorname{card} I = k}} B_I$$

and ψ induces a well-defined linear surjection

$$G^k(\mathbb{R}^N)/E \to B_J/B_J \cap \sum_{\substack{I \subset J \\ \operatorname{card} I = k}} B_I .$$

For each $l \in \mathbb{N}^{N-k}$ with $|l| = k$, let

$$\mu_l = \frac{\partial^k}{\partial y_{k+1}^{l_1} \cdots \partial y_N^{l_{N-k}}} \lambda_1 \wedge \cdots \wedge \lambda_k \in G^k(\mathbb{R}^N).$$

The representatives of $\{\mu_l \text{ s.t. } l \in \mathbb{N}^{N-k}, |l| = k\}$ generate $G^k(\mathbb{R}^N)/E$, so

$$\dim B_J/B_J \cap \sum_{\substack{I \subset J \\ \operatorname{card} I = k}} B_I \leqslant \binom{k + (N-k) - 1}{k} = \binom{N-1}{k}.$$

Since there are $\binom{K+1}{k+1}$ subsets of $\{0,\ldots, K\}$ which have cardinality $k+1$,

$$\dim \sum_{\substack{J \subset \{0,\ldots,K\} \\ \operatorname{card} J \leqslant k+1}} B_J \leqslant \binom{K+1}{k+1}\binom{N-1}{k} + \dim \sum_{\substack{I \subset \{0,\ldots,K\} \\ \operatorname{card} I \leqslant k}} B_I .$$

Therefore,

$$\dim \sum_{J \subset \{0,\ldots,K\}} B_J \leqslant \sum_{k=0}^{K} \binom{K+1}{k+1}\binom{N-1}{k},$$

using induction starting at $\dim \sum_{\operatorname{card} J \leqslant 0} B_J = 0$.

This finishes the proof since

$$\sum_{k=0}^{K} \binom{K+1}{k+1}\binom{N-1}{k} = \binom{K+N}{N},$$

as is well known; see, e.g., [9, p. 822].

The following lemma is used to prove property (3.3) of χ.

LEMMA 4.3. *Let $N \in \mathbb{N}^+$, $K \in \mathbb{N}$, and $x_0, \ldots, x_K \in \mathbb{R}^N$ independent. For $J \subset \{0,\ldots, K\}$, let B_J be as defined in Lemma 4.2, above.*

Let $g \in \mathscr{C}^K(\mathbb{R}^N)$ such that $T(g) = 0$ for every $T \in \sum_{J \subset \{0,\ldots,K\}} B_J$.

Then for every $k \in \{0,\ldots, K\}$, every $q \in Q^k(\mathbb{R}^N)$ and every $J \subset \{0,\ldots, K\}$ with card $J = k+1$, *there exists $x \in [x_j]_{j \in J}$ such that $q(\partial/\partial x)\, g(x) = 0$.*

Proof. Let $C \subset \mathbb{R}^N$ be the linear span of $\{x_j - x_i \text{ s.t. } i, j \in J\}$, let $y_1, \ldots, y_k$

be a basis of C, complete it to a basis $y_1, \ldots, y_k, y_{k+1}, \ldots, y_N$ of $\mathbb{R}^N$, and let $\lambda_1, \ldots, \lambda_N \in \mathbb{R}_N$ be the dual basis. Let $\mu = q(\partial/\partial x)\,\lambda_1 \wedge \cdots \wedge \lambda_k \in G^k(\mathbb{R}^N)$.

Choose an orientation for $[x_j]_{j\in J}$ and define

$$T\colon \mathscr{C}^K(\mathbb{R}^N) \to \mathbb{R} \text{ by } T(f) = \int_{[x_j]_{j\in J}} \phi_k(\mu, f),$$

where ϕ_k is defined in Lemma 4.2. Then $T \in B_J$.

By hypothesis,

$$T(g) = \int_{[x_j]_{j\in J}} q\left(\frac{\partial}{\partial x}\right) g(x)\,\lambda_1 \wedge \cdots \wedge \lambda_k = 0.$$

Since $[x_j]_{j\in J}$ is connected, $q(\partial/\partial x)\,g(x)$ is continuous, and $\lambda_1 \wedge \cdots \wedge \lambda_k$ is a determinant function on C, there exists $x \in [x_j]_{j\in J}$ such that $q(\partial/\partial x) \times g(x) = 0$.

COROLLARY 4.4. *Let $p \in P^K(\mathbb{R}^N)$ such that $T(p) = 0$ for every $T \in \sum_{J\subset\{0,\ldots,K\}} B_J$. Then p is the zero polynomial.*

Proof. By contradiction. Suppose $\deg p = k \in \{0,\ldots, K\}$. Then for every $i \in \mathbb{N}^N$ with $|i| = k$, there exists $x \in [x_0, \ldots, x_k]$ such that $(\partial^{|i|}/\partial x^i)\,p(x) = 0$, so $\deg p \leqslant k - 1$.

COROLLARY 4.5. $\dim \sum_{J\subset\{0,\ldots,K\}} B_J = \binom{K+N}{K}$.

Proof. $\dim P^K(\mathbb{R}^N) = \binom{K+N}{K}$.

Remark 4.6. For $N \in \mathbb{N}^+$, $K \in \mathbb{N}$ and $x_0, \ldots, x_K \in \mathbb{R}^N$ independent, Lemmas 4.2 and 4.3 imply the existence of χ in Theorem 3.1. To see this, for all $J \subset \{0,\ldots, K\}$, let $B_J \subset \mathscr{D}^K(\mathbb{R}^N)$ be the set defined in Lemma 4.2 and let $\chi\colon \mathscr{C}^K(\mathbb{R}^N) \to P^K(\mathbb{R}^N)$ be given by the property that $T(\chi(f) - f) = 0$ for all $f \in \mathscr{C}^K(\mathbb{R}^N)$ and all $T \in \sum_{J\subset\{0,\ldots,K\}} B_J$. χ is well-defined and linear because of the duality between $\sum_{J\subset\{0,\ldots,K\}} B_J$ and $P^K(\mathbb{R}^N)$, proven in Lemma 4.2 and Corollary 4.4.

Then by Lemma 4.3, for every $f \in \mathscr{C}^K(\mathbb{R}^N)$, every $k \in \{0,\ldots, K\}$, every $q(\partial/\partial x) \in Q^k(\mathbb{R}^N)$, and every $J \subset \{0,\ldots, K\}$ with card $J = k + 1$, there exists $x \in [x_j]_{j\in J}$ such that $q(\partial/\partial x)(\chi(f) - f)(x) = 0$.

Proof of Theorem 3.1. It only remains to generalize the preceding remark to the case where $x_0, \ldots, x_K$ are not necessarily distinct. For $k \in \{0,\ldots, K\}$, let $y_k = (x_k, 0,\ldots, 0, 1, 0,\ldots, 0) \in \mathbb{R}^{N+K+1}$, where the unit is in the $N + k + 1$st place. Let $\pi\colon \mathbb{R}^{N+K+1} \to \mathbb{R}^N$ be the projection onto the first N coordinates and let $\pi^*\colon \mathscr{C}^K(\mathbb{R}^N) \to \mathscr{C}^K(\mathbb{R}^{N+K+1})$ be given by $\pi^*(f) = f \circ \pi$.

Since $y_0, \ldots, y_K$ are independent, let $\psi\colon \mathscr{C}^K(\mathbb{R}^{N+K+1}) \to P^K(\mathbb{R}^{N+K+1})$ be the map whose existence is proven in Remark 4.6 and define $\chi\colon \mathscr{C}^K(\mathbb{R}^N) \to P^K(\mathbb{R}^N)$

by $\pi^* \circ \chi = \psi \circ \pi^*$. χ is well defined because for $f \in \mathscr{C}^K(\mathbb{R}^N)$, $\pi^*(f) \in \mathscr{C}^K(\mathbb{R}^{N+K+1})$ is independent of the last $K+1$ variables. By Proposition 3.7 (in case $K \geqslant 1$), so is $\psi \circ \pi^*(f)$. That is, $\psi \circ \pi^*(f)$ is a polynomial of degree at most K, in the first N coordinates of $\mathbb{R}^{N+K+1}$ only.

Now choose $f \in \mathscr{C}^K(\mathbb{R}^N)$, $q(\partial/\partial x) \in Q^k(\mathbb{R}^N)$, where $k \in \{0,..., K\}$, and let $J \subset \{0,..., K\}$ with card $J = k+1$. By Remark 4.6, let $y \in [y_j]_{j \in J}$ such that

$$q\left(\frac{\partial}{\partial x}\right) \pi^* \circ \chi(f)(y) = q\left(\frac{\partial}{\partial x}\right) \psi \circ \pi^*(f)(y)$$
$$= q\left(\frac{\partial}{\partial x}\right) \pi^*(f)(y)$$

Then $q(\partial/\partial x)(\chi(f) - f)(\pi(y)) = 0$ and $\pi(y) \in [x_j]_{j \in J}$ because $\pi(y_j) = x_j$ for $j \in \{0,..., K\}$.

Remark 4.7. Lemma 4.3 can be generalized to a statement about finite families of functions and differential operators. Let S be a finite indexing set and suppose $\{g_s\}_{s \in S} \subset \mathscr{C}^K(\mathbb{R}^N)$ such that $T(g_s) = 0$ for every $s \in S$ and every $T \in \sum_{J \subset \{0,...,K\}} B_J$. ($B_J$ has been defined only when $x_0, ..., x_K$ are independent.) Then, for every $k \in \{0,..., K\}$, every indexed set $\{q_s\}_{s \in S} \subset Q^k(\mathbb{R}^N)$, and every $J \subset \{0,..., K\}$ with card $J = k+1$, there exists $x \in [x_j]_{j \in J}$ such that $\sum_{s \in S} q_s(\partial/\partial x)\, g_s(x) = 0$.

Therefore, if $\{f_s\}_{s \in S} \subset \mathscr{C}^K(\mathbb{R}^N)$, then there exists $x \in [x_j]_{j \in J}$ such that $\sum_{s \in S} q_s(\partial/\partial x)(\chi(f_s) - f_s)(x) = 0$. The method of the preceding proof of Theorem 3.1 can be used to prove this when $x_0, ..., x_K$ are not necessarily distinct.

As in Proposition 3.7, if $\sum_{s \in S} q_s(\partial/\partial x)(f_s)$ is identically zero, so is $\sum_{s \in S} q_s(\partial/\partial x)(\chi(f_s))$.

Remark 4.8. The proof of Lemma 4.2 shows that, if $x_0, ..., x_K$ are in general position in $\mathbb{R}^N$ (that is, every subset of $\{x_0, ..., x_K\}$ of cardinality $N+1$ is independent), then $\chi(f)$ is well defined for $f \in \mathscr{C}^H(\mathbb{R}^N)$, where $H = \text{minimum}\ \{K, N-1\}$. This corresponds to the fact that the Hermite interpolating polynomial $\chi(f)$, at distinct points $x_0, ..., x_K \in \mathbb{R}$, is defined for $f \in \mathscr{C}^0(\mathbb{R})$. In this case χ is, of course, Lagrange interpolation.

Remark 4.9. We will sometimes use the notation $\chi_{x_0 \ldots x_K}$ to indicate the dependence of χ on the points of interpolation. For a fixed $f \in \mathscr{C}^K(\mathbb{R}^N)$, consider the symmetric mapping from $(\mathbb{R}^N)^{K+1}$ to $P^K(\mathbb{R}^N)$ which is given by $x_0, ..., x_K \mapsto \chi_{x_0 \ldots x_K}(f)$. The techniques used here can also be used to show that this mapping is continuous.

EXAMPLE 4.10. The conclusion of Theorem 3.1 cannot be strengthened to include non-homogeneous differential operators. For, let $K = 1$, $x_0 = (0, 0)$, and $x_1 = (1, 0) \in \mathbb{R}^2$. Define $f(x, y) \in \mathscr{C}^1(\mathbb{R}^2)$ by $f(x, y) =$

$(6x^2 - 6x + 1)\, y - 6x^2 + 6x$. $\chi(f)$ is the zero polynomial. Yet if $q(\partial/\partial x, \partial/\partial y)$ is the non-homogeneous operator $f \mapsto \partial f/\partial y + f$, $q(\partial/\partial x, \partial/\partial y) f(x, 0) = 1$ for all $x \in [0, 1]$.

5.

This section gives versions of χ for interpolating complex analytic functions and differential forms. $\mathbb{C}^N$ is identified with $\mathbb{R}^{2N}$ and a point $(z_1, \ldots, z_N) \in \mathbb{C}^N$ has real and imaginary components $(x_1, \ldots, x_N, y_1, \ldots, y_N)$.

PROPOSITION 5.1. *Let $N \in \mathbb{N}^+$, $K \in \mathbb{N}$, $z_0, \ldots, z_K \in \mathbb{C}^N$, and let χ be the interpolation at $(x_0, y_0), \ldots, (x_K, y_K) \in \mathbb{R}^{2N}$ given in Theorem* 3.1.

For $h: \mathbb{C}^N \to \mathbb{C}$, K times continuously differentiable, let $\chi(h) = \chi(f) + i\chi(g)$, where $h = f + ig$, real and imaginary parts.

If h is analytic, so is $\chi(h)$, that is, $\chi(h) \in P^K(\mathbb{C}^N)$.

Proof. If $K = 0$, $\chi(f)$ and $\chi(g)$ are constants, so we may assume $K \in \mathbb{N}^+$. Then by Remark 4.7, since $f + ig$ satisfies the Cauchy–Riemann equations, $\partial f/\partial x_n - \partial g/\partial y_n \equiv 0$ and $\partial f/\partial y_n + \partial g/\partial x_n \equiv 0$ for all $n \in \{1, \ldots, N\}$, so does $\chi(f) + i\chi(g)$.

Remark 5.2. For a fixed analytic $h: \mathbb{C}^N \to \mathbb{C}$, the mapping from $(\mathbb{C}^N)^{K+1} \to P^K(\mathbb{C})$, which is given by $z_0, \ldots, z_K \mapsto \chi_{z_0 \ldots z_K}(h)$ is continuous because its real and imaginary parts are continuous. See Remark 4.9.

Remark 5.3. Consider the case where $N = 1$ and $w_0, \ldots, w_K$ are distinct complex numbers. If $h: \mathbb{C} \to \mathbb{C}$ is analytic, $\chi_{w_0 \ldots w_K}(h) \in P^K(\mathbb{C})$ and $\chi_{w_0 \ldots w_K}(h)(w_k) = h(w_k)$ for $k \in \{0, \ldots, K\}$. As in the corresponding real case (Lagrange interpolation) these properties uniquely determine $\chi_{w_0 \ldots w_K}(h)$.

Remark 5.4. The complex analytic analog of Proposition 3.9 holds. Let χ be the complex analytic interpolation at $z_0, \ldots, z_K \in \mathbb{C}^N$. If $\lambda \in \mathbb{C}_N$, the dual of $\mathbb{C}^N$, and $h: \mathbb{C} \to \mathbb{C}$ is analytic, then $\chi(h \circ \lambda) = \psi(h) \circ \lambda$, where $\psi(h)$ is the complex analytic interpolation of h at the points $\lambda(z_0), \ldots, \lambda(z_K) \in \mathbb{C}$. We omit a proof of this; it is similar to the proof of Proposition 3.9.

Remark 5.5. We define an interpolation of differential forms. Notation is explained in Remark 4.1. Let $N \in \mathbb{N}^+$, $K \in \mathbb{N}$, $x_0, \ldots, x_K \in \mathbb{R}^N$, and let χ be the interpolation of Theorem 3.1 at $x_0, \ldots, x_K$. For $n \in \mathbb{N}$, let $\chi: A^{n,K}(\mathbb{R}^N) \to \{\omega \in A^{n,0}(\mathbb{R}^N)$ s.t. ω is a polynomial of degree $\leqslant K\}$ be given by

$$\chi\Big(\sum_{1 \leqslant s_1 < \cdots < s_n \leqslant N} f_{s_1 \cdots s_n} \lambda_{s_1} \wedge \cdots \wedge \lambda_{s_n}\Big) = \sum_{1 \leqslant s_1 < \cdots < s_n \leqslant N} \chi(f_{s_1 \cdots s_n})\, \lambda_{s_1} \wedge \cdots \wedge \lambda_{s_n}.$$

Remark 5.6. If ω is a closed form ($d\omega$ is identically zero), then so is

$\chi(\omega)$, by a proof which uses Remark 4.7 in a manner similar to the proof of Proposition 5.1. However, if $\omega \in A^{n,K}(\mathbb{R}^N)$, then $d\omega \in A^{n+1,K-1}(\mathbb{R}^N)$ so $\chi(d\omega)$ is not defined in general and the statement "$\chi(d\omega) = d\chi(\omega)$" is false.

6.

An application of χ is given. Consider a sequence $\{{}_cT\}_{c\in\mathbb{N}}$ of distributions in $\mathscr{D}^0(\mathbb{R}^N)$, such that card supp ${}_cT \leqslant K+1$ for all c and $\lim_{c\to\infty} \max_{x\in\text{supp}_c T}\{|x|\} = 0$. where $\text{supp}_c T$ is the support of ${}_cT$. That is, $\{{}_cT\}$ is a sequence of linear combinations of $\leqslant K+1$ Dirac measures whose supports tend to the origin. Let $E \subset \mathscr{C}^K(\mathbb{R}^N)$ be the solution set of some differential equation $q(\partial/\partial x)(f) \equiv 0$, where q is a homogeneous operator and suppose $\lim_{c\to\infty}\{{}_cT(p)\}$ exists for every $p \in E \cap P^K(\mathbb{R}^N)$. Under these hypotheses, the following theorem says that $\lim_{c\to\infty}\{{}_cT(f)\}$ exists for every $f \in E$.

THEOREM 6.1. *Let $N \in \mathbb{N}^+$ and $K \in \mathbb{N}$. Let $\{{}_cT\}_{c\in\mathbb{N}}$ be a sequence of distributions in $\mathscr{D}^K(\mathbb{R}^N)$, each having finite support. For each $c \in \mathbb{N}$, let ${}_cJ = -1 +$ card $\text{supp}_c T$ and let $\text{supp}_c T = \{{}_c y_0, \dots, {}_c y_{{}_cJ}\} \subset \mathbb{R}^N$. For each $c \in \mathbb{N}$ and $j \in \{0,\dots,{}_cJ\}$, let ${}_cM_j$ be the order of ${}_cT$ at ${}_c y_j$, and suppose that $\sum_{j=0}^{{}_cJ}({}_cM_j + 1) = K+1$ for all c and $\lim_{c\to\infty} \max_{j\in\{0,\dots,{}_cJ\}} |{}_c y_j| = 0$.*

Explicitly for each $c \in \mathbb{N}$, there exists a finite indexed set of real numbers

$$\{{}_c a_{jm}\}_{j\in\{0,\dots,{}_cJ\}\, m\in\mathbb{N}^N \text{s.t. } |m| \leqslant {}_cM_j}$$

such that

$${}_cT(f) = \sum_{j=0}^{{}_cJ} \sum_{\substack{m\in\mathbb{N}^N \\ |m|\leqslant {}_cM_j}} {}_c a_{jm} \frac{\partial^{|m|}}{\partial x^m} f({}_c y_j)$$

for all $f \in \mathscr{C}^K(\mathbb{R}^N)$.

Let F be any linear subspace of $\bigoplus_{k=0}^K Q^k(\mathbb{R}^N)$ such that, if $q(\partial/\partial x) \in F$, then each homogeneous part of q is also in F, and let

$$E = \left\{ f \in \mathscr{C}^K(\mathbb{R}^N) \text{ s.t. } q\left(\frac{\partial}{\partial x}\right)(f) \equiv 0 \textit{ for every } q \in F \right\}.$$

Suppose that $\{{}_cT(p)\}_{c\in\mathbb{N}}$ converges for every $p \in E \cap P^K(\mathbb{R}^N)$. Then, for each $f \in E$, $\{{}_cT(f)\}_{c\in\mathbb{N}}$ converges to $\lim_{c\to\infty}\{{}_cT(\chi_0(f))\}$ where $\chi_0(f)$ is the Taylor polynomial of f up to order K at the origin.

Proof. Since F is generated by its homogeneous elements, $\chi_0(f) \in E$ and $\lim_{c\to\infty}\{{}_cT(\chi_0(f))\}$ exists by the hypothesis on $\{{}_cT\}$.

For each $c \in \mathbb{N}$, choose ${}_cx_0, \dots, {}_cx_K \in \mathbb{R}^N$ such that, for each $j \in \{0,\dots,{}_cJ\}$, ${}_cx_k = {}_c y_j$ for ${}_cM_j + 1$ values of $k \in \{0,\dots,K\}$. This is possible because

$\sum_{j=0}^{cJ} ({}_cM_j + 1) = K + 1$. For each $c \in \mathbb{N}$, let ${}_c\chi$ be the interpolation of Theorem 3.1 at ${}_cx_0, \ldots, {}_cx_K$.

Given $\epsilon > 0$, it suffices to find $C \in \mathbb{N}$ so large that $| {}_cT(f) - T(\chi_0(f)| \leqslant \epsilon$ for all $c \geqslant C$, where $T(\chi_0(f)) = \lim_{c \to \infty} \{{}_cT(\chi_0(f))\}$. Fix $L > 0$ so large that, for each $c \in \mathbb{N}$ and each $p \in E \cap P^K(\mathbb{R}^N)$ having the absolute value of all coefficients $\leqslant 1$, $| {}_cT(p)| \leqslant L$.

By Remark 4.9, the mapping from $(\mathbb{R}^N)^{K+1}$ to $P^K(\mathbb{R}^N)$ given by $y_0, \ldots, y_K \mapsto \chi_{y_0, \ldots, y_K}(2Lf/\epsilon)$ is continuous. Choose $C \in \mathbb{N}$ so large that, for each $c \geqslant C$, $({}_c\chi - \chi_0)(2Lf/\epsilon)$ has the absolute value of each of its coefficients $\leqslant 1$, and also, for $c \geqslant C$, $| {}_cT(\chi_0(f)) - T(\chi_0(f)| \leqslant \epsilon/2$.

For all c, ${}_cT(f) = {}_cT({}_c\chi(f))$ because $f - {}_c\chi(f)$ is flat of order ${}_cM_j$ at ${}_cy_j$ for $j \in \{0, \ldots, {}_cJ\}$ (Remark 3.4). Therefore, for $c \geqslant C$,

$$\begin{aligned} | {}_cT(f) - T(\chi_0(f))| &\leqslant | {}_cT({}_c\chi - \chi_0)(f)| \\ &\quad + | {}_cT(\chi_0(f)) - T(\chi_0(f))| \\ &\leqslant \frac{\epsilon}{2} + \frac{\epsilon}{2} \end{aligned}$$

because $({}_c\chi - \chi_0)(2Lf/\epsilon)$ has the absolute value of each coefficient $\leqslant 1$ and also belongs to E, by Proposition 3.7, so

$$\left| {}_cT({}_c\chi - \chi_0)\left(\frac{2Lf}{\epsilon}\right)\right| \leqslant L$$

from the definition of L.

Remark 6.2. The special case of the preceeding theorem for $F = \{0\}$, $E = \mathscr{C}^K(\mathbb{R}^N)$ was essentially given by Glaeser in [4]. Bloom [1] gives the complex analytic version, which may also be proven using Proposition 5.1. $\chi(f)$ is the same as the interpolation used there whenever $f \in P^{K+1}(\mathbb{C}^N)$. In that case, the method used in [1] to interpolate f coincides with the method given here in the next section.

7.

Proposition 7.3 can be used to find $\chi(f)$ without evaluating integrals, whenever f is a polynomial.

Remark 7.1. Let $W_0, \ldots, W_K$ be indeterminates. For $M \in \mathbb{N}$, there exist unique polynomials.

$$\sigma_0(W_0, \ldots, W_K), \ldots, \sigma_K(W_0, \ldots, W_K) \in \mathbb{R}[W_0, \ldots, W_K],$$

the ring of polynomials with real coefficients in $K + 1$ indeterminants, such that

$$W_k{}^M = \sum_{l=0}^{K} \sigma_l(W_0, \ldots, W_k) \cdot W_k{}^l \qquad \text{for } k \in \{0, \ldots, K\}. \tag{7.2}$$

For uniqueness, if $\tau_0, \ldots, \tau_K$ is another such family of polynomials, then by Cramer's rule,

$$(\sigma_l - \tau_l)(W_0, \ldots, W_K) \begin{vmatrix} 1 & W_0 & \cdots & W_0{}^K \\ \vdots & \vdots & & \vdots \\ 1 & W_K & \cdots & W_K{}^K \end{vmatrix} = 0$$

for $l \in \{0, \ldots, K\}$ and the van der Monde determinant is not zero. For existence, if $M \in \{0, \ldots, K\}$, $\sigma_M = 1$ and $\sigma_l = 0$ if $l \neq M$. If existence is proven up to $M \geqslant K$, then the formula

$$W_k^{M+1} = \sum_{l=0}^{K} \tau_l(W_0, \ldots, W_K) \cdot W_k^{M-l} \qquad \text{for } k \in \{0, \ldots, K\}$$

gives an inductive step, where $-\tau_l$ is the elementary symmetric polynomial of degree $l + 1$.

If real numbers are substituted for $W_0, \ldots, W_K$, then $\sigma_l(x_0, \ldots, x_K)$ is the coefficient of x^l in the Hermite interpolating polynomial at $x_0, \ldots, x_K$ for the function $f \in \mathscr{C}^K(\mathbb{R})$ given by $f(x) = x^M$.

PROPOSITION 7.3. *Let $N \in \mathbb{N}^+$, $K \in \mathbb{N}$, and $x_0, \ldots, x_K \in \mathbb{R}^N$. Also, let $M \in \mathbb{N}$ and define $\sigma_0, \ldots, \sigma_K \in \mathbb{R}[W_0, \ldots, W_K]$ by (7.2). Then, for each $m \in \mathbb{N}^N$ with $|m| = M$,*

$$\chi_{x_0, \ldots, x_K}\left(\frac{M!}{m!} x^m\right) = \sum_{\substack{k \in \mathbb{N}^N \\ M-K \leqslant |k| \\ k \leqslant m}} \frac{|m-k|!}{(m-k)!} \alpha_k x^{m-k},$$

where α_k is the coefficient of V^k in

$$\sigma_{M-|k|}(x_{01}V_1 + \cdots + x_{0N}V_N, \ldots, x_{K1}V_1 + \cdots + x_{KN}V_N).$$

($V_1, \ldots, V_N$ are indeterminates, $V^k = V_1^{k_1} \cdot \cdots \cdot V_N^{k_N}$ and the components of x_k are $(x_{k1}, \ldots, x_{kN})$ for $k \in \{0, \ldots, K\}$. $\chi_{x_0, \ldots, x_K}$ is interpolation at $x_0, \ldots, x_K$.)

Proof. Let $p: \mathbb{R} \to \mathbb{R}$ be given by $p(w) = w^M$, so for every $(\lambda_1, \ldots, \lambda_N) = \lambda \in \mathbb{R}_N$ and every $(y_1, \ldots, y_N) = y \in \mathbb{R}^N$, $p \circ \lambda(y) = \sum_{m \in \mathbb{N}^N\, |m| \leqslant M} (M!/m!)\, y^m \lambda^m$.

By Proposition 3.9,

$$\chi_{x_0\cdots x_K}(p\circ\lambda)(y)$$
$$=\chi_{\lambda(x_0)\cdots\lambda(x_K)}(p)(\lambda(y)) \tag{7.4}$$
$$=\sum_{l=0}^{K}\sigma_l(x_{01}\lambda_1+\cdots+x_{0N}\lambda_N,\ldots,x_{K1}\lambda_1+\cdots+{}_{KN}\lambda_N)(y_1\lambda_1+\cdots+y_N\lambda_N)^l.$$

Here, $\chi_{\lambda(x_0)\ldots\lambda(x_K)}$ is interpolation in $\mathbb{R}^1$ at $\lambda(x_0),\ldots,\lambda(x_K)$. For any fixed $m\in\mathbb{N}^N$ with $|m|=M$ and each $k\in\mathbb{N}^N$ such that $M-K\leqslant|k|$ and $k\leqslant m$, the coefficient of y^{m-k} in the right-hand side of (7.4) is

$$\sigma_{M-|k|}(x_{01}\lambda_1+\cdots+x_{0N}\lambda_N,\ldots,x_{K1}\lambda_1+\cdots+x_{KN}\lambda_N)\cdot\frac{|m-k|!}{(m-k)!}\lambda^{m-k}$$
$$=\sum_{\substack{j\in\mathbb{N}^N\\|j|=|k|}}\alpha_j\frac{|m-k|!}{(m-k)!}\lambda^{m-k+j}.$$

because $\sigma_{M-|k|}$ is homogeneous of degree $|k|$. Hence the right-hand side of (7.4) is

$$\sum_{\substack{k\in\mathbb{N}^N\\M-K\leqslant|k|\\k\leqslant m}}\left(\sum_{\substack{j\in\mathbb{N}^N\\|j|=|k|}}\alpha_j\frac{|m-k|!}{(m-k)!}\lambda^{m-k+j}\right)y^{m-k}. \tag{7.5}$$

Now, let $S=\binom{M+N-1}{M}$ and choose $\lambda_1,\ldots,\lambda_S\in\mathbb{R}_N$ and $a_1,\ldots,a_S\in\mathbb{R}$ such that $\sum_{s=1}^{S}a_s p\circ\lambda_s(y)=(M!/m!)\,y^m$ for every $y\in\mathbb{R}^N$. That is, for $j\in\mathbb{N}^N$ with $|j|=M$, $\sum_{s=1}^{S}a_s\lambda_s^j=1$ if $j=m$ and equals zero otherwise. Then by linearity of $\chi_{x_0\ldots x_K}$ and applying (7.5),

$$\chi_{x_0\cdots x_K}\left(\frac{M!}{m!}x^m\right)(y)=\sum_{\substack{k\in\mathbb{N}^N\\M-K\leqslant|k|\\k\leqslant m}}\alpha_k\frac{|m-k|!}{(m-k)!}y^{m-k}$$

for every $y\in\mathbb{R}^N$.

Remark 7.6. The complex version of Proposition 7.3 can be stated and proven by writing $\mathbb{C}$ rather than $\mathbb{R}$ and referring to Remark 5.4 rather than Proposition 3.9.

Acknowledgments

The contents of this paper appear in my thesis at the University of Toronto. I am grateful for the encouragement of my supervisor, Thomas Bloom, who conjectured Theorem 6.1. I also had many helpful discussions with Ed Bierstone and Pierre Milman.

References

1. T. Bloom, On the Zariski tangent space to a complex analytic variety, *Math. Ann.* **214** (1975), 159–166.
2. C. de Boor, Polynomial interpolation, Proc., Int. Cong. Helsinki (1978), in press.
3. H. Flanders, "Differential Forms," Academic Press, New York, 1963.
4. G. Glaeser, L'interpolation des fonctions différentiables de plusieurs variables, Proc. Liverpool Singularities—Symposium II, Lecture Notes in Mathematics No. 209, Springer-Verlag, Berlin/Heidelberg/New York, 1971.
5. P. Kergin, "Interpolation of C^K Functions," Thesis, University of Toronto, 1978.
6. C. Micchelli and P. Milman, A formula for Kergin interpolation in R^k, *J. Approximation Theory* **29** (1980), 294–296.
7. F. Treves, "Topological Vector Spaces, Distributions, and Kernels," Academic Press, New York, 1967.
8. B. Wendroff, "Theoretical Numerical Analysis," Academic Press, New York, 1966.
9. M. Abramowitz and A. Stegun, "Handbook of Mathematical Functions," Dover, New York, 1965.